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# VYSHNEGRADSKIY'S PROBLEM IN THE THEORY OF PRIMARY REGULATION: PART I. THEORY OF THE PRIMARY-ACTION REGULATOR WHEN COULOMB AND VISCOUS FRICTION ARE PRESENT

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[Figures are appended.]

Abstract: The basic problem of the theory of primary regulation was examined, taking into account both viscous and coulomb friction. By utilizing Lyapunov's second method and the theory of point transformation of a surface, we examined the qualitative structure of the "pulse" portions of the corresponding phase space on the trajectory. We also found some quantitative characteristics of this phase space which are of interest to the profession.

The results of research were summarized in diagrams, which permit a solution of problems of convergence or nonconvergence in regulation processes.

In the present work on regulation systems dependent on Vyshnegradskiy's equations (it should be pointed out that the theory of many recent regulating apparatus can be reduced with sufficient accuracy to Vyshnegradskiy's problem, in its complete form as well as in that particular form corresponding to the absence of viscous friction -- see example 14) the nature of motion arising during various initial conditions -- that is, the qualitative structure of "the pulse" of the phase space on the trajectory -- is ascertained and diagrams are given which enable the rapid determination of convergence or nonconvergence of regulation processes for given parameters and given load variations. As far as the authors know, the study of Vyshnegradskiy's problems as problems of analytical dynamics or as problems in the theory of nonlinear vibration is given here for the first time. In the past such a study was not satisfactorily conducted, even for the special case of the absence of viscous friction.



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Present research in problems connected with clarifying the qualitative aspects of the "pulse" of the phase space on the trajectory relies, to a great degree, on the theory of point transformations which in the beginning was connected with the works of Poincare, Brouwer, Birkhoff. These problems evidently are destined to play an important role in not only celestial and terrestrial mechanics, particularly in automatic regulation, but also in some studies connected with the case of discontinuity under study, the so-called second method of Lapyunov, which enables many problems involving stability to be solved with a minimum of calculations. The diagrams in this work are generalizations of the earlier diagrams of Vyshnegradskiy and Mises. Some of the curves plotted on these diagrams possessed comparatively simple analytical expressions. Other curves are plotted approximately by a graphic procedure like that of Mises.

The brief account of results contained in the present work is published in the form of two notes in DAN (Reports of the Academy of Sciences USSR) [15, 16].

## I. THE EQUATIONS OF MOTION; STATEMENT OF THE PROBLEM

Let us examine an engine (for example, a turbine) equipped with the usual centrifugal regulator and let us assume that a rigid connection exists between the regulator clutch displacement and the valve opening that regulates steam input. Let us also assume that the shifting of this valve is done by the regulator itself without the application of external energy.

The kinetic equations of such a dynamic system (engine equipped with a primary-action regulator) subject to ordinary simplifications (these simplifications, introduced by I. A. Vyshnegradskiy, are: 1) disregard all nonlinearity except that due to coulomb friction; 2) disregard the engine's automatic regulation; 3) disregard the gyroscopic term in the main shaft's equation of rotation; 4) assume that the engine load varies arbitrarily and that the equations describe the system's behavior after any unexpected variation in load, and hence the clutch displacement and also the variation in the angular velocity are regarded as known in a new steady-state system) and written in ordinary engineering symbols and with allowance for viscous and coulomb friction (the latter introduces nonlinearity) can be written for $\frac{d\xi}{d\tau} \neq 0$ in the form:

$$\left.\begin{aligned} m\frac{d^2\xi}{d\tau^2} &= -\frac{2E\delta}{a}\xi + \frac{2E}{\omega m}\eta - \rho\frac{d\xi}{d\tau} \mp K \\ J\frac{d\eta}{d\tau} &= -\frac{M}{a}\xi, \end{aligned}\right\} \quad (A)$$

where

$\tau$ -- time,

$\xi$ -- clutch displacement figured from the new position of equilibrium (See, for example, references 13, 17, and 18 in the bibliography. After the proper selection of generalized forces, the equation (A) can be written in the form of Lagrangian equations of the second type.)

$\eta$ -- deviation of the angular velocity from its new constant value (see note under "$\xi$" above)

m -- mass of regulator reduced to the coordinate of the clutch,

a -- distance moved by the clutch regulator,

E -- reduced force or so-called "energy" of the regulator,

$\rho$ -- coefficient of viscous friction reduced to clutch movement,



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K -- absolute magnitude of coulomb (rigid) frictional force reduced to clutch movement,

$\omega_m$ -- average angular velocity of engine,

$\delta = \frac{\omega_2 - \omega_1}{\omega_m}$ -- the regulator's so-called coefficient of irregularity, where $\omega_2$ is the angular velocity, corresponding to the highest position of the clutch and $\omega_1$ is the angular velocity, corresponding to the lowest position,

J -- the engine's reduced moment of inertia,

M -- average torque of the engine.

Initial conditions, as usual, can be given in the form:

$$\xi = -\lambda a, \ \left(\frac{d\xi}{d\tau}\right)_0 = 0, \quad \eta = -\omega_m \delta \lambda, \tag{A'}$$

where $\lambda$ (during release of the engine load, $0<\lambda<1$) represents the relative variation of the engine load; that is, it is the ratio of sudden variation in load to full load.

The separate terms of the right part of equation A possess the following mechanical significance: $\frac{2E\delta}{a}\xi$ is the so-called adjustable force due to the incorrect (that is, not conforming to the stationary system) position of the regulator's clutch; $+\frac{2E}{\omega_m}\eta$ is the so-called adjustable force due to the incorrect value of the engine's angular velocity; $-\rho_0\frac{d\xi}{d\tau}$ is the viscous friction force reduced to the clutch moment; $+$ K is the coulomb frictional force reduced to the clutch movement; $-\frac{M}{a}\xi$ is the so-called excess moment that acts on the engine's main shaft with the force of the clutch displacement.

Now we will introduce dimensionless coordinates $\xi_1 = \frac{\xi}{a}$ and $\eta_1 = \frac{\eta}{\omega_m}$ and new parameters $T_a$, $T_r$, $T_k$, having dimensions of time: $T_a = I_a/M$ is the so-called time of setting the engine in motion;
$T_r = \sqrt{\frac{am}{2E}}$ is the so-called half-time of free fall of the clutch;
$T_k = \frac{\rho_a}{2E}$ is the so-called half-time characterizing the stroke regulator (viscous friction); and dimensionless parameter $\varepsilon = \frac{K}{E}$ called coefficient of regulator insensitivity. Then the system (A) takes on the form:

$$\left.\begin{aligned} T_r^2\frac{d^2\xi_1}{d\tau^2} + T_k\frac{d\xi_1}{d\tau} + \delta\xi_1 &= \eta_1 \mp \frac{\varepsilon}{2}, \\ T_a\frac{d\eta_1}{d\tau} &= -\xi_1, \end{aligned}\right\} \tag{B}$$

and the initial conditions (A') take the formula:

$$\xi_{10} = -\lambda, \ \left(\frac{d\xi_1}{d\tau}\right)_0 = 0, \quad \eta_{10} = -\delta\lambda. \tag{B'}$$

From the point of view of the theory of vibration we deal here with an unusual oscillating system, in which the usual harmonic oscillator, possessing both viscous and coulomb friction, is connected with the simplest rotor in such a way that the oscillator displacement influences the input of external energy in a rotary degree of freedom, while the deviation from equilibrium of the rotor's angular velocity produces forces that act upon the oscillator. (It must be kept in mind that in setting up the second equation in (A) we disregarded the gyroscopic term (related to the term $+2E/\omega_m\eta$ in the first equation) and the term characterizing the self-compensation of the engine. If we had taken these terms into consideration but omitted term $-M/a\xi$ in the second equation, we would have obtained an elementary dissipative gyroscopically connected oscillator-rotor system. The presence of term

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$-M/a\ \xi$ in the second equation, we would have obtained an elementary dissipative gyroscopically connected oscillator-rotor system. The presence of term $-M/a\ \xi$ in the right-hand part of the second equation in (A) causes our system (for some region of parameter values) to develop possible instability.

System (B) contains five parameters $T_a$, $T_r$, $T_k$, $\delta$, $\varepsilon$. Let us make the substitution

$$\xi_1 = \varepsilon \sqrt[3]{\frac{T_a^2}{T_r^2}} \cdot x, \quad \eta_1 = \varepsilon y, \quad \tau \sqrt[3]{T_r^2 T_a} \cdot t,$$

and thereby the transition to system (C), which contains only two independent dimensionless parameters $A = \delta \sqrt[3]{\frac{T_a^2}{T_r^2}}$ and $B = T_k/T_r \sqrt[3]{T_a/T_r}$

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + B\frac{dx}{dt} + Ax &= y \mp \frac{1}{2}, \\ \frac{dy}{dt} &= -x. \end{aligned}\right\} \qquad \text{(C)}$$

We will notice that parameters A and B, which are called the main parameters of the theory of primary regulation, or Vyshnegradskiy's parameters, do not contain quantities that characterize coulomb friction. This means that, in the variables chosen by us, the "pulse" of the phase space on the trajectory does not depend upon the magnitude of the force of the coulomb friction. By no means does this mean that the type of motion arising for these or other load variations does not depend upon the coulomb friction, since the variation in engine load in our problem is directly characterized by the initial position of the representative point in the phase space and since the initial conditions for system (C) are connected with the quantity $\lambda$, which characterizes load variation, according to the relations,

$$x_0 = -\frac{\lambda}{\varepsilon} \sqrt[3]{\frac{T_r^2}{T_a^2}}; \quad \left(\frac{dx}{dt}\right)_0 = 0, \quad y_0 = -\frac{\delta}{\varepsilon}\lambda, \qquad \text{(C')}$$

into which the coefficient $\varepsilon$ of insensitivity enters. If we take into consideration the force of coulomb friction and we determine the system of equation (C) for the case $dx/dt = 0$ and represent it in the form of three simple first-degree equations, then we arrive at the system

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = -x, \quad \frac{dz}{dt} = F(x, y, z), \qquad \text{(D)}$$

where

$$F(x,y,z) = \begin{cases} -Ax + y - Bz - \frac{1}{2}, & \text{if } z > 0 \text{ or if } z = 0 \text{ and } -Ax + y > \frac{1}{2}, \\ -Ax + y - Bz + \frac{1}{2}, & \text{if } z < 0 \text{ or if } z = 0 \text{ and } -Ax + y < -\frac{1}{2}, \\ 0, & \text{if } -\frac{1}{2} \leqslant -Ax + y \leqslant +\frac{1}{2},\ z = 0. \end{cases}$$

System (D) and the necessary conditions governing continuity of the functions $x(t)$, $z(t)$ at the points of discontinuity in $F(x, y, z)$ determine the study of the dynamic problem in question.

In the case of the usual centrifugal regulator examined by us, the initial conditions in accordance with (C') can be expressed in the form:

$$x = x_0, \quad y = A x_0, \quad z = \frac{dx}{d\tau} = 0. \tag{D'}$$

(For some other arrangements of automatic regulation, particularly in the case of so-called "inertia" regulators, it is necessary to deal with more general initial conditions.)

In regard to the dynamic problem (D) the present work contains: 1) qualitative study of the structure of the "pulse" of the phase space on the trajectory for various values of the parameters A and B; 2) the quantitative study for various values of the parameters A and B in the region of stability in the large interval of rest $x = z = 0$, $-1/2 \leq y \leq +1/2$ (each point of this interval ($x = z = 0$, $-1/2 \leq y \leq +1/2$) represents a state of equilibrium for system (D)) applicable to initial conditions (D').

## II. SUMMARY OF RESULTS RELATING TO A LINEAR CASE

In order to simplify the following exposition we will give a brief summary of some results relating to the case where the coulomb friction is absent. Assuming in system (B) $\epsilon = 0$ and substituting

$$\xi = \sqrt[3]{\frac{T_a^2}{T_r^2}}\, x, \quad y = y, \quad \tau = \sqrt[3]{T_r^2 T_a}\cdot t,$$

we obtain the linear system:

$$\frac{dx}{d\tau} = z, \quad \frac{dy}{d\tau} = -x, \quad \frac{dz}{d\tau} = -Ax + y - Bz \tag{E}$$

with the same parameters A and B as in the case of system (D). The only state of equilibrium in system (E) is at the origin of the coordinates.

The characteristic equation of system (E) has the form:

$$r^3 + Br^2 + Ar + 1 = 0 \tag{1}$$

The roots of equation (1) will be negative or will have negative real parts if $AB > 1$. If however, $AB < 1$, then among the roots there will be roots with positive real parts.

Let us examine the space of the parameters of system (E) by laying off along the rectangular axes the parameters $A > 0$ and $B > 0$ (Figure 1). Then the curve $W_1$ with equation $AB = 1$ (so-called Vyshnegradskiy's hyperbola) will divide the region of those values of (A) and (B) that correspond to stability (region U: $AB > 1$) from the region of those values of (A) and (B) that correspond to instability (region V: $AB < 1$). Later we will be interested in more details of the case $AB < 1$. In this case the real root of equation (1) is negative, and the two complex conjugate roots possess positive real parts, that is

$$r_1 = s, \; r_2 = p + jq, \; r_3 = p - jq \; (s < 0, \; p > 0, \; q > 0). \tag{2}$$

If we assume that $\quad \frac{s}{q} = -a \; (a > 0), \; \frac{p}{q} = b \; (b > 0),$

then we can express, in the region V, the roots of equation (1) and the parameters A and B by means of these new and very convenient parameters a and b:

$$p = \frac{b}{\sqrt[3]{a(1+b^2)}}, \quad q = \frac{1}{\sqrt[3]{a(1+b^2)}}, \quad s = -\frac{a}{\sqrt[3]{a(1+b^2)}} \tag{3}$$

$$A = \frac{1+b^2-2ab}{\sqrt[3]{a^2(1+b^2)^2}}, \quad B = \frac{a-2b}{\sqrt[3]{a(1+b^2)}} \tag{4}$$

Conditions $A>0$, $B>0$, $AB<1$ become respectively conditions $1+b^2-2ab\geqslant 0$, $a\geqslant 2b$, $b>0$. With this, hyperbola $AB=1$ on the A, B -plane transform into straight line $b=0$ (a-axis) on the a, b- plane. (See figure 1a.)

Let us study, finally, some properties of the trajectories of system (E), which we will have to use later.

1. The trajectories of system (E), intersecting plane $z=0$, increase in time magnitude t from negative values of z to positive values (upward) the half plane in surface $y>Ax$ and vice versa (downward) into half plane $y<Ax$. On points of the straight line $y-Ax$ the trajectories touch the plane $z=0$, whereupon near the point of contact for $x>0$ they remain under the plane $z=0$, but for $x<0$ they are above it. These properties of the trajectories are directly deduced from the equations in (E).

2. If the values of the parameters of system (E) lie in the region V, then the representative point, if it does not move along one of the two critical trajectories, certainly intersects plane $z=0$.

Actually the general solution of system (E) can be written in the form:

$$x=k_1 C e^{s\tau}+m_1 C_2 e^{p\tau}\cos q\tau+l_1 C_3 e^{p\tau}\sin q\tau,$$
$$y=k_2 C_1 e^{s\tau}+m_2 C_2 e^{p\tau}\cos q\tau+l_2 C_3 e^{p\tau}\sin q\tau,$$
$$z=C_1 e^{s\tau}+C_2 e^{p\tau}\cos q\tau+C_3 e^{p\tau}\sin q\tau,$$

where $C_1$, $C_2$, $C_3$, are constants of integration and $k_1$, $k_2$, $m_1$, $m_2$, $l_1$, $l_2$ are certain constants that are independent of initial conditions and are determinable from system (E).

If the values of parameters A and B lie in region V, then $s<0$ and $p>0$. Therefore only if $C_2^2+C_3^2\neq 0$, then z changes its sign and moreover changes an infinite number of times as time t approaches positive infinity. If however, $C_2=C_3=0$, then z for all values of time τ retains one and the same sign as that of $C_1$. Giving $C_1$ various values, we obtain one and the same trajectory for all $C_1>0$ and another trajectory for all C $C_1<0$. Disregarding these two extra-ordinary trajectories, we can consider from the analysis of case $AB<1$ that the initial point $(x_0, y_0, z_0)$ for $t=0$ lies in the plane x, y; that is, that $z_0=0$. It should be noted that in the case $AB<1$ such an assertion is not always right. If all three roots of equation (1) are real and negative, then the trajectories, exceptional only in the sense just indicated, form a three-dimensional continuum.

Since system (D) contains the same parameters A and B as system (E), we can retain during the analysis of system (D) the same space of coefficients A and B. However, the structure of this space of coefficients will now be essentially different. Region U ($AB>1$), as we shall see in the next paragraph, will again be the region of stability; while region V for system (D) will have to be divided into two parts; region $V_1$, the so-called region of conditional stability and region $V_2$, the so-called field of absolute instability (see paragraphs 5 and 6).

## III. REGION OF ABSOLUTE STABILITY U ($AB>1$)

In the case of the linear system (E) for the condition $AB>1$, all three roots of the characteristic equation (1) possess negative real parts, and the region of attraction of the equilibrium state $x=y=z=0$ embraces all phase space. We will show that for $AB>1$, rigid (coulomb) friction does not disrupt stability and that in this case the region of attraction of the interval of rest of the system (D) also embraces all phase space. (It is not difficult to point out problems from the theory of regulation, where the

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presence of rigid friction produces instability). Or, in other words, we will prove the theorem: if $AB>1$, then in system (D) the representative point for any initial conditions unconditionally approaches with increase in time t the interval of rest $x = z = 0$, $-1/2 \leqslant y \leqslant +1/2$. The proof of this theorem is based on the study of Laypunov's functions with some modifications of it applicable to the case of discontinuity under discussion.

Let us examine first of all the family of triaxial ellipsoids:

$$V(x, y, z) \equiv [-1 + \alpha(B + A^2)]x^2 + \alpha y^2 + (A\alpha - B)z^2 - 2A\alpha xy + \\ + 2(A + \alpha)^2 xz - 2yz = C, \qquad (5)$$

where $\alpha$ is a positive constant greater than $\frac{A + B^2}{AB - 1}$ (for example it is possible to set $\alpha = 1 + \frac{A + B^2}{AB - 1}$). During adherence to the indicated condition, the surfaces of (5) actually represent ellipsoids which easily can be checked on the basis of usual criteria.

It is easy to establish the following properties of this family of surfaces:

1. Ellipses $V(x, y, 0) = C$ have the straight line $y = Ax$ as their diameter, joined by a chord parallel to axis y.

Hence it follows that for any point $(x_0, y_0, 0)$ the following inequality holds:

$$V(x_0, y_0, 0) > V(x_0, Ax_0, 0). \qquad (6)$$

2. During the movement of the representative point along the trajectory of the linear system (E) the following relation holds:

$$\frac{dV}{dt} = -2(Ax - y)^2 - [\alpha(AB - 1) - B^2 - A]z^2 \leqslant 0. \qquad (7)$$

From this it follows that if the representative point at any moment of time $t = t_0$ was located on the ellipsoid $V = C_0$, then for the time $t = t_1 > t_0$ it falls on ellipsoid $V = C_1$, where $C_1 < C_0$. In this way, the family of ellipsoids (5) reveals that for system (E) under the condition $AB>1$, the whole space is the region of attraction of the equilibrium state (0, 0, 0).

Now we will pass on to system (D). The trajectories of system (D) can be obtained from the trajectories of system (E) by the following geometric construction.

1. If for the points of the examined part of the trajectory of system (D) we have $z > 0$ or $z = 0$, $-Ax + y - 1/2 > 0$, or, finally, $z = 0$, $-Ax + y - 1/2 = 0$, $x < 0$, then this part of the trajectory of system (D) can be obtained from the corresponding part of the trajectory of (E) by displacing the latter by the amount $+1/2$ in the direction of the y axis. Therefore, for the study of the behavior of such parts of the trajectory of system (D), the upper halves of the ellipsoids of (5) can be used, by displacing them also by $+1/2$ in the direction of the y axis. This family of displaced upper halves of the ellipsoids of (5) we will denote in this way:

$$V_{+\frac{1}{2}}(x, y, z) = C, \qquad (5')$$

and in place of (6) we will now have:

$$V_{+\frac{1}{2}}(x_0, y_0, 0) > V_{+\frac{1}{2}}(x_0, Ax_0 + \tfrac{1}{2}, 0). \qquad (6')$$

2. If for the points of the examined part of a trajectory of system (D) we have $z<0$ or $z = 0$, $-Ax + y - 1/2 < 0$, or, finally, $z = 0$, $-Ax + y + 1/2 = 0$, $x>0$, then this part of a trajectory of system (D) can be obtained from the corresponding part of a trajectory of (E) by displacing the latter by $-1/2$ in the direction of the y axis. Therefore, for the study of the behavior

of such parts of a trajectory of system (D), the lower halves of the ellipsoids of (5) can be used, by displacing them by $-1/2$ in the direction of the $y$ axis. This family of displaced lower halves of the ellipsoids of (5) we will denote in this way:

$$V_{-\frac{1}{2}}(x, y, z) = C, \tag{5''}$$

from which it is obvious that

$$V_{-\frac{1}{2}}(x_0, y_0, 0) = V_{+\frac{1}{2}}(x_0, y_0 + 1, 0). \tag{8}$$

3. Finally, if for the points of the examined part of the trajectory (D) we have $z = 0$, $-1/2 < -Ax + y < +1/2$ or $z = 0$, $-Ax + y + 1/2 = 0$, $x < 0$, or $z = 0$, $-Ax + y - 1/2 = 0$, $x > 0$, then this part of the trajectory of system (D) is a segment of the straight line parallel to the $y$ axis. The representative point moves along this segment when $x > 0$ on the side of decreasing $y$ and when $x < 0$, on the side of the increasing $y$ until it falls either on the half line $-Ax + y + 1/2 = 0$, $x > 0$, or on the half line $-Ax + y - 1/2 = 0$, $z = 0$, $x < 0$. Further motion is executed in conformity with case 1 when $x < 0$ and case 2 when $x > 0$. Ellipses $V_{+\frac{1}{2}}(x, y, 0) = C$, ellipses $V_{-\frac{1}{2}}(x, y, 0) = C$ and the motion on the segments, parallel to the $y$ axis, are shown in Figure 2.

We will not go on to prove the theorem. First of all we note that equations (D) do not change with the substitution of $x$, $y$, $z$ by $-x$, $-y$, $-z$, respectively.

Let the initial position of the representative point (for the time $t = t_0$) be $x_0$, $y_0$, $z_0$. By virtue of what has just been stated we can consider without limiting the generality that $z_0 \geqslant 0$. If $z \neq 0$ for time $t_0 < t < +\infty$, we will have with an accuracy up to a displacement of 1/2 along the y-axis, ~~the same movement as for the linear case. By virtue of (7), in this case,~~ the representative point approaches in the limit $(0, \pm 1/2, 0)$ (the sign is plus if $z_0 > 0$) and it is minus if $z_0 < 0$).

In order to prove our theorem, we only have to prove that if the representative point is incident twice upon the plane $z = 0$, then it starts its movement in space on an ellipse of lower number, after the second arrival on this plane, rather than after the first arrival on the plane.

Let first the representative point fall on the plane $z = 0$ (on the point $x_0$, $y_0$, 0). Without limiting the generality it can be considered that it will start further to move in the upper half space and consequently, from ellipsoid $V_{+\frac{1}{2}}(x_0, y_0, C) = C_0$. During this movement, it will transfer, by virtue of (7) on to ellipsoid $V_{+\frac{1}{2}}(x, y, z) = C$, where $C < C_0$ and $C$ decreases with increasing time $t$. Therefore, if the representative point for time $t = t_1$ falls on the plane $z = 0$ (on point $x_1$, $y_1$, 0), then:

$$C_1 = V_{+\frac{1}{2}}(x_1, y_1, 0) < C_0.$$

By virtue of system (D) for $x_1$, $y_1$, we have the inequality (as $\left(\frac{dz}{dt}\right)_{t=t'} \leqslant 0$):

$$-Ax_1 + y_1 - \frac{1}{2} \leqslant 0.$$

Furthermore, 4 cases are possible:

(1) $x_1 = 0$, $-1/2 < y_1 < 1/2$. Then movement will cease and the representative point in a finite interval of time will reach the segment of rest.

(2) $x_1 > 0$, $y_1 \leqslant + Ax_1 - 1/2$ or $x_1 < 0$, $y_1 < Ax_1 - 1/2$.

Then by virtue of system (D) the representative point must continue its movement under the plane $z = 0$ and, consequently, instead of the family $V_{+\frac{1}{2}} = C$ we must now take the family $V_{-\frac{1}{2}} = \bar{C}$.

By virtue of (8) and (6') we have:

$$\bar{C}_1 = V_{-\frac{1}{2}}(x_1, y_1, 0) = V_{+\frac{1}{2}}(x_1, y_1 + 1, 0) < V_{+\frac{1}{2}}(x_1, y_1, 0) = C_1 < C_0$$

(3) $x_1 > 0$, $Ax_1 - 1/2 < y_1 \leqslant Ax_1 + 1/2$. Then the representative point as was noted earlier, must move along the straight line $x = x_1$, until it falls on point $x_1$, $\bar{y}_1 = Ax_1 - 1/2$, and continues its movement in the space under the x, y plane.

But once again by virtue of (8) and (6') we have:

$$\bar{C}_1 = V_{-\frac{1}{2}}(x_1, \bar{y}_1, 0) = V_{-\frac{1}{2}}\left(x_1, Ax_1 - \tfrac{1}{2}, 0\right) < V_{+\frac{1}{2}}\left(x_1, Ax_1 + \tfrac{1}{2}, 0\right) <$$
$$< V_{+\frac{1}{2}}(x_1, y_1, 0) = C_1 < C_0.$$

(4) $x_1 < 0$, $Ax_1 - 1/2 \leqslant y < Ax_1 + 1/2$. Then the representative point must move along the straight line $x = x_1$, until it falls on point $x_1$, $\bar{y}_1 = Ax_1 + 1/2$, and continues its movement under the x, y plane.

By virtue of (6')

$$\bar{C}_1 = V_{+\frac{1}{2}}(x_1, \bar{y}_1, 0) = V_{+\frac{1}{2}}\left(x_1, Ax_1 + \tfrac{1}{2}, 0\right) < V_{+\frac{1}{2}}(x_1, y_1, 0) = C_1.$$

In this way our assertions are proved in all cases.

Speaking of system (D), we will call the region $U(AB > 1)$ the region of absolute stability in the space of parameters A, B. The region of conditional stability in the parameter space will be called the region of such values of parameters A, B for which parameters in the phase space of system (D) the region of extension of the segment of rest does not embrace the entire phase space.

## IV. THE REGION V ($AB < 1$); REDUCTION OF THE DYNAMIC PROBLEM TO POINT TRANSFORMATION; UNDERREGULATION AND OVERREGULATION

By virtue of the observation made in Section II, in the case of $AB < 1$ the representative on any (except the two extraordinary trajectories, the existence of which can be disregarded, keeping in mind the unavoidable fluctuations) trajectory of system (D) reaches the plane $z = 0$ in a finite interval of time. Therefore, during an examination of the movement of the representative point in the phase space we can accept that $z = 0$ when the time $t = 0$. It is obvious that, in the examined case, the studying of the behavior of each separate trajectory in the phase space is equivalent to studying the succession of points of intersection with the plane $z = 0$, and the studying of the structure of the "pulse" of the phase space on the trajectory is equivalent to studying the structure of the transformation of the plane $z = 0$ into itself, which (transformation) is effected by the movement of the representative point on the trajectory.

For convenience of further calculations we will make one more substitution of variables in system (D), by setting (do not confuse with those $\xi$, $\eta$, $\tau$ which figures Sections I and II):

$$\tau = q t, \quad \xi = \frac{x}{q}, \quad \eta = y, \quad \zeta = \frac{z}{q^2}. \qquad (9)$$

Then (D) transform into (D'')

$$\frac{d\xi}{d\tau} = \zeta, \quad \frac{d\eta}{d\tau} = -\xi, \quad \frac{d\zeta}{d\tau} = \frac{1}{q^3} F(\xi, \eta, \zeta), \qquad (D'')$$

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where

$$F(\xi,\eta,\zeta)\equiv\begin{cases}-Aq\xi+\eta-Bq^2\zeta-\frac{1}{2} & \text{with } \zeta>0 \text{ and with } \zeta=0 \text{ and } -Aq\xi+\eta-\frac{1}{2}>0,\\ 0 & \text{with } \zeta=0,\ -\frac{1}{2}\leqslant -Aq\xi+\eta\leqslant+\frac{1}{2},\\ -Aq\xi+\eta-Bq^2\zeta+\frac{1}{2} & \text{with } \zeta<0 \text{ and with } \zeta=0 \text{ and } -Aq\xi+\eta+\frac{1}{2}<0,\end{cases}$$

whereupon, in accordance with (3), we will have:

$$Aq=\frac{1+b^2-2ab}{a(1+b^2)},\quad Bq^2=\frac{a-2b}{a(1+b^2)} \qquad (10)$$

Let us divide the plane $\xi,\eta$ ($\zeta=0$) into three sectors (see Figure 3):

$$\left.\begin{array}{l}G^{(1)}, \text{ where } -Aq\xi+\eta-\frac{1}{2}>0 \text{ or } -Aq\xi+\eta-\frac{1}{2}=0,\ \xi<0\\ G^{(2)} \text{ where } -\frac{1}{2}<-Aq\xi+\eta<+\frac{1}{2} \text{ or } -Aq\xi+\eta-\frac{1}{2}=0,\ \xi>0,\\ \qquad \text{or } -Aq\xi+\eta+\frac{1}{2}=0,\ \xi<0,\\ G^{(3)} \text{ where } Aq\xi+\eta+\frac{1}{2}<0 \text{ or } -Aq\xi+\eta+\frac{1}{2}=0,\ \xi>0.\end{array}\right\} \qquad (11)$$

Let the initial point $(\xi_0, \eta_0, 0)$ corresponding to $\tau=0$, belong $G^{(2)}$. Then the solution of (D'') will have the form:

$$\xi=\xi_0,\quad \eta=-\xi_0\tau,\quad \zeta=0. \qquad (12)$$

The representative point moves according to rule (12) until it leaves sector $G^{(2)}$, transferring when $\xi_0>0$ into sector $G^{(3)}$, and when $\xi_0<0$ into sector $G^{(1)}$. At this moment its coordinates, according to (11) and (12), will be

$$\xi_1=\xi_0,\quad \eta_1=Aq\xi_0-\frac{1}{2}\,\mathrm{sign}\,\xi_0,\quad \zeta_1=0. \qquad (13)$$

Let us agree to call the above-described transfer of point $(\xi_0, \eta_0, 0)$ into point $(\xi_1, \eta_1, 0)$ the transformation E. Transformation E, in this way, is single-valued (unique) and continuous, but not reciprocally unique.

Now let the initial point $(\xi_0, \eta_0, 0)$, corresponding to $\tau=0$, belong to sector $G^{(1)}$. By virtue of (D'') the representative point starts its movement in the upper half space ($\zeta>0$) and again returns to the plane $\zeta=0$ to point $(\xi_1, \eta_1, 0)$ at the moment of time $\tau=t$.

By virtue of (D'') it is easy to write in parametric form the formula giving the transition from $\xi_0, \eta_0$ to $\xi_1, \eta_1$ by introducing the parameters t, the time of transition, and v, a certain auxiliary parameter: the movement of the representative point in the examined case, by virtue of the equations in (D'') is expressed thus:

$$\left.\begin{array}{l}\xi=C_1ae^{-a\tau}-C_2e^{b\tau}[b\cos\tau-\sin\tau]-C_3e^{b\tau}[\cos\tau+b\sin\tau],\\ \eta-\frac{1}{2}=C_1e^{-a\tau}+C_2e^{b\tau}\cos\tau+C_3e^{b\tau}\sin\tau,\\ \zeta=-C_1a^2e^{-a\tau}-C_2e^{b\tau}[(b^2-1)\cos\tau-2b\sin\tau]-C_3e^{b\tau}\\ \qquad [2b\cos\tau+(b^2-1)\sin\tau],\end{array}\right\} \text{(a)}$$

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whereupon, since $\zeta_0 = 0$, we have:

$$C_1 a^2 + C_2 (b^2 - 1) + C_3 2b = 0. \tag{b}$$

The value of t is determined as the least positive root of the equation:

$$-C_1 a^2 e^{-at} - C_2 e^{bt}[(b^2-1)\cos t - 2b\sin t] - C_3 e^{bt}[2b\cos t + (b^2-1)\sin t] = 0. \tag{c}$$

According to (a) and (b), $C_1$, $C_2$, $C_3$ can be determined through $\xi_0, \eta_0$, and then equation (c) will give the transition time t. However, it is much more convenient to do it in another way: express $C_1$, $C_2$, $C_3$ with time as a parameter and in the same way also express $\xi_0, \eta_0$ as functions of t and of a second auxiliary parameter v, by setting:

$$\left.\begin{aligned} C_1 &= v\left[\frac{(1+b^2)^2}{a^2}\sin t\right], \\ C_2 &= v\left[2be^{-(a+b)t} - 2b\cos t - (b^2-1)\sin t\right], \\ C_3 &= v\left[-(b^2-1)e^{-(a+b)t} + (b^2-1)\cos t - 2b\sin t\right] \end{aligned}\right\} \tag{d}$$

If in (a), we set $\tau = 0$, and substitute in $C_1$, $C_2$, $C_3$ their expressions from (d), then we will obtain $\xi_0, \eta_0$ as functions of v and t. If in (a), setting $\tau = t$, we again substitute in $C_1$, $C_2$, $C_3$ their expressions from (d), then we will obtain $\xi_1, \eta_1$ as functions of v and t. This will give formulas (14) and (15) presented in the text.

$$\left.\begin{aligned} \xi_0 &= v f(t), & \eta_0 - \tfrac{1}{2} &= v g(t), \\ \xi_1 &= -v e^{-at} f(-t), & \eta_1 - \tfrac{1}{2} &= -v e^{-at} g(-t), \end{aligned}\right\} \tag{14}$$

where:

$$\left.\begin{aligned} f(t) &= (1+b^2)\left[\cos t + \frac{[1 - b(a+b)]}{a}\sin t - e^{-(a+b)t}\right], \\ g(t) &= -2b\cos t + \left[\frac{(1+b^2)^2 - a^2(b^2-1)}{a^2}\right]\sin t + 2be^{-(a+b)t}. \end{aligned}\right\} \tag{15}$$

In this way (14) and (15) give through parameters v and t the rule for transforming the point $\xi_0, \eta_0$ of plane $\zeta = 0$ to the point $\xi_1, \eta_1$ of the same plane. It remains only to determine the interval of variation of the parameters v and t. We will start with finding the interval of variation of parameter t. We will notice that, according to (14), all points with the same "transition time" t lie on the ray which passes through the extremity of the segment of rest (0, 1/2) and has an angular coefficient

$$k(t) = \frac{g(t)}{f(t)}. \tag{16}$$

The transformed points $(\xi_1, \eta_1)$ lie on the ray which passes through the same point (0, 1/2) and have an angular coefficient (see Figure 3)

$$k(-t) = \frac{g(-t)}{f(-t)}. \tag{17}$$

It is easy to show that

1. $$\lim_{t\to 0} k(t) = \lim_{t \to 0} k(-t) = Aq. \tag{18}$$

In this way the points with a zero "transition time" lie on the boundary of sectors $G^{(1)}$ and $G^{(2)}$, that is, on the ray $-Aq\,\xi + \eta - 1/2 = 0$, $\xi > 0$. This follows directly from the note made in Section II that along this ray the trajectories of the linear system (taking into consideration obviously the 1/2 displacement in the direction of the $\eta$-axis) touch the plane, remaining under this plane near the point of contact.

2. $$k'(t)=(1+b^2)\frac{1+(a+b)^2}{a^2}e^{-(a+b)t}\{e^{(a+b)t}-\cos t-(a+b)\sin t\} \qquad (19)$$

When $t > 0$ in our case, since $a > 0$ and $b > 0$, we will have: $k'(t) > 0$. The function of $k$ (5) is a monotonically increasing function of $t$. In this way the ray, carrying points with "transition time" $t$, turns with increasing time $t$ to a positive direction (counterclockwise).

We will "sweep," during variation in time $t$, the entire $G^{(1)}$ sector by this ray (all the while passing through each of its points only once), if we will keep varying $t$ all the while until we come to the other boundary of $G^{(1)}$ and $G^{(2)}$; that is, to the ray $-Aq\xi+\eta-1/2=0; \xi<0$. In this way the upper boundary of the variation in time $t$ will be obtained as the least positive root of the equation

$$k(t) = Aq,$$

or, in a developed form after a simple transformation, of the equation:

$$e^{-(a+b)t}-\cos t+(a+b)\sin t=0. \qquad (20)$$

Denoting the smallest positive root of this equation by $\theta = \theta(a+b)$, we thus obtain $t$'s interval of variation

$$0 \leqslant t \leqslant \theta \qquad (21)$$

The values of the function $\theta(a+b)$ are given in the following table:

| $a+b$ | $\theta$ | $a+b$ | $\theta$ | $a+b$ | $\theta$ |
|---|---|---|---|---|---|
| 0 | $2\pi$ | 0.601 | 4,239 | 1,312 | 3.794 |
| 0.023 | 5.758 | 0.660 | 4.182 | 1.378 | 3.773 |
| 0.030 | 5.685 | 0.720 | 4.130 | 1.439 | 3.752 |
| 0.060 | 5.456 | 0.744 | 4.111 | 1.500 | 3.732 |
| 0.120 | 5.257 | 0.750 | 4.106 | 1.559 | 3.714 |
| 0.180 | 4.950 | 0.780 | 4.083 | 1.621 | 3.696 |
| 0.240 | 4.790 | 0.840 | 4.040 | 1.680 | 3.680 |
| 0.2745 | $\frac{3}{2}\pi$ | 0.900 | 4.000 | 1.733 | 3.656 |
| 0.300 | 4.660 | 0.960 | 3.964 | 2.673 | 3.500 |
| 0.330 | 4.603 | 1.020 | 3.930 | 3.790 | 3.400 |
| 0.360 | 4.551 | 1.080 | 3.899 | 6.278 | 3.330 |
| 0.420 | 4.457 | 1.140 | 3.870 | ⋮ | ⋮ |
| 0.480 | 4.376 | 1.200 | 3.843 | ⋮ | ⋮ |
| 0.541 | 4.303 | 1.260 | 3.818 | $\infty$ | $\pi$ |

It is not difficult to see that as $(a+b) \to +\infty$, $\theta \to \pi$.

From the above table it is seen that $\theta$ is a decreasing function of $(a+b)$. Its graph is given in Figure 4.

We will notice that from the expression for $k'(t)$ derived above, it follows that when $0 < t \leqslant \theta$ we will have:

$$\frac{d}{dt}\{k(-t)\} < 0.$$

Consequently, the function $k(-t)$ monotonically decreases, and the ray

$$\xi_1 = -ve^{-at}f(-t), \qquad \eta_1 - \frac{1}{2} = -ve^{-at}g(-t)$$

turns clockwise, with increase in t, from the initital position corresponding to $t = 0$ to some finite value corresponding to $t = \theta$. Thus the ray "sweeps" part of the sectors $G^{(2)}$ and $G^{(3)}$.

Let us now examine v's interval of variation. It is determined by the fact that the points of the ray $\xi_0 = vf(t)$, $\eta_0 - 1/2 = vg(t)$ must belong to sector $G^{(1)}$. Since for $v = 0$ we obtain the initial point of ray $(0, +1/2)$, then the whole problem is reduced to determing the sign of v.

For small t's, it is quite easy to see that

$$f(t) = (1+b^2)\left\{t\,\frac{[1+(a+b)^2]}{a} + \dots\right\},$$

since for small t's we have $\xi_0 > 0$, then $v > 0$; therefore,

$$0 \leqslant v < +\infty \tag{22}$$

In this way each point $(\xi_0, \eta_0)$ of sector $G^{(1)}$ plane $\zeta = 0$, moving in accordance with (D'') in the half space $\xi > 0$ during the time t, falls on point $\xi_1$, $\eta_1$ of the same plane belonging to $G^{(2)}$ or $G^{(3)}$; whereupon the points with one and the same transition time t lie on one and the same ray:

$$\xi_0 = vf(t), \quad \eta_0 - \frac{1}{2} = vg(t), \quad v > 0$$

and fall on one and the same ray

$$\xi_1 = -ve^{-at}f(-t), \quad \eta_1 - \frac{1}{2} = -ve^{-at}g(-t), \quad v > 0.$$

Parameter t determines the angular coefficient of the ray and varies from 0 to $\theta$; parameter v determines the position of the point on the ray and varies from 0 to $+\infty$.

Let us call the previously described transformation of points $G^{(1)}$ into points of $G^{(2)}$ and $G^{(3)}$ transformation $S^{+}$. This transformation is reciprocally unique (single-valued), by virtue of the fact that the trajectories of the linear system (E) in space do not intersect, and is continuous.

Completely analogous reasoning, calculations, and formulas determine the transition of the points of $G^{(3)}$ into the points of $G^{(1)}$ and $G^{(2)}$, since system (D'') does not change with the substitution of $\xi, \eta, \zeta$ by $-\xi, -\eta, -\zeta$, whereupon the sector $G^{(2)}$ passes into itself. Instead of (14) we will now have

$$\left.\begin{aligned} &\xi_0 = vf(t), \quad \eta_0 + \frac{1}{2} = vg(t), \\ &\xi_1 = -ve^{-at}f(-t), \quad \eta_1 + \frac{1}{2} = -ve^{-at}g(-t) \end{aligned}\right\} \tag{14'}$$

where the value of v now must be taken negative: $-\infty < v < 0$. The transformation of the points of $G^{(1)}$ and $G^{(2)}$, described in formula (14'), we will call transformation S- It goes without saying that the transformation also is reciprocally unique (single-valued) and continuous.

To shorten further exposition let us introduce symbolic designations for the original rays and the rays transformed by transformations S+ and S$^-$. Ray $\{\xi = vf(t), \eta = 1/2 + vg(t), v > 0\}$ we will denote by symbol $\lambda(t)$; and the ray obtained from $\lambda(t)$ by transformation S+, that is, the ray $\{\xi = -vf(-t)e^{-at}, \eta = 1/2 \cdot vg(-t)e^{-at}, v > 0\}$ will be expressed by the symbol $\lambda_1(t)$. Ray $\{\xi = vf(t), \eta = -1/2 + vg(t), v < 0\}$ we will denote by the symbol $\lambda'(t)$; and the ray obtained from $\lambda'(5)$ by transformation S$^-$, that is, the ray $\{\xi = -vf(-t)e^{-at}, \eta = -1/2 - vg(-t)e^{-at}, v < 0\}$ will be expressed by the symbol $\lambda_1'(t)$ (see Figure 3).

We will agree for the future to distinguish the so-called case of underregulation from the case of overregulation.

Let the initial values $\xi_0, \eta_0$ be such that $-Aq\,\xi_0 + \eta_0 - 1/2 = 0$ and $\xi_0 < 0$; that is, points $\xi_0, \eta_0$ lie on ray $\lambda(\theta)$.

We will say that underregulation takes place if for the transformed points $\xi_1, \eta_1$ we have $\xi_1 < 0$, and we will say that overregulation takes place if for the transformed points we have $\xi_1 > 0$. The limit or boundary between cases of underregulation and overregulation is determined by the discontinuity in $k(-\theta)$. The expression for $k(-\theta) = \dfrac{g(-\theta)}{f(-\theta)}$ easily can be changed to the following form by using (20):

$$k(-\theta) = \frac{1}{a(1+b^2)} \cdot \frac{[a+b+b^3-ab^2-(1-2ab+b^2)\cot\theta]}{(b-\cot\theta)} - Aq + \frac{1}{b-\cot\theta}.$$

From this it is seen that for the limiting case

$$b = \cot\theta.$$

Equations (23) and (20) enable one to find a and b corresponding to this limit and, through them, A and B. The corresponding values of A and B are given in the following table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | 2.623 | 2.476 | 2.260 | 2.161 | 2.210 | 2.467 | 2.681 | 2.747 | 3.000 |
| B | 0 | 0.225 | 0.689 | 1.185 | 1.718 | 2.313 | 2.647 | 2.724 | 3.000 |

Curves corresponding to this limit are charted in Figures 1 and 1a as the line R. The region of underregulation lies to the right of this curve. From the expression for $k(-\theta)$ it is seen that in the case of underregulation $b > \cot\theta$ and in the case of overregulation $b < \cot\theta$.

Let us return to the transformation of the plane $\zeta = 0$. Every point of the plane $\zeta = 0$ depending upon its location is either not displaced (points of the segment of rest $\xi = 0$, $-1/2 \leqslant \eta < +1/2$) or undergoes one of three transformations: S$^+$, S$^-$ or E.

Performing the corresponding transformation S$^+$, S$^-$ or E for every point of the plane, we obtain the transformation of the entire plane $\zeta = 0$ within itself, which we denote by T and which is unique (single valued) but not reciprocally unique (because of the points of sector $G^{(2)}$ which are subject to transformation E).

In this way, in the examined case (AB<1) our dynamic problem (D") is reduced to point transformation T of the plane $\zeta = 0$ to itself.

## V. REGION V (AB<1); STUDY OF THE STABILITY OF THE REST SEGMENT FOR SMALL DISPLACEMENTS

Let us examine the behavior (during a repetition of transformation T) of points lying sufficiently close to the fixed points of transformation that form the segment of rest ($\xi = 0, -1/2 \leq \eta + 1/2$). Since the points of $G^{(2)}$ according to transformation E pass into the points of $G^{(1)}$ or $G^{(3)}$, then by virtue of symmetry it is sufficient to examine the points of $G^{(1)}$ close to the point $(0, 1/2)$. Insofar as, for points of $G^{(1)}$ close to $(0, 1/2)$, the parameter v has small values, the transformed points after transformation E will fall on ray $\lambda(\theta)$ or $\lambda'(\theta)$. Since transformation E does not change the abscissae of the transformed points, then upon the repetition of transformation T the points will approach in the limit the segment of rest if for sufficiently small $\xi$, we have

$$|\xi_0| > |\xi_1|. \tag{24}$$

With the opposite inequality they will, at least with the first repetition of T, withdraw from the segment of rest; not one point of plane $\xi, \eta$ will then approach the segment of rest, as to its own limit, during repetition of transformation T. Condition (24) is reduced to

$$|f(\theta)| > e^{-a\theta}|f(-\theta)|. \tag{24'}$$

It is not difficult to verify that for the case of underregulation this condition is reduced to the inequality $e^{-b\theta} > -b\sin\theta + \cos\theta$ and, as it is easy to see, by virtue of (20) it is always fulfilled. In the case of overregulation the condition (24') is reduced to the inequality

$$e^{-b\theta} > b\sin\theta - \cos\theta \tag{25}$$

and fulfilled for only some region of the values a, b and, consequently, also A, B. The parametric representation of the limit of that region of the A, B plane where the segment of rest is stable (curve $W_2$, Figure 5), can be written in the form

$$A = \frac{1 + b^2 - 2ab}{\sqrt[3]{a^2(1+b^2)^2}}; \quad B = \frac{a - 2b}{\sqrt[3]{a(1+b^2)}} \tag{26}$$

where a and b are functions of the auxiliary parameter $\theta$ in agreement with the implicit equations:

$$e^{-b\theta} = -\cos\theta + b\sin\theta, \quad e^{-(a+b)\theta} = \cos\theta - (a+b)\sin\theta. \tag{26'}$$

The values of A, B for this curve $W_2$ are given in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | 1.261 | 1.250 | 1.227 | 1.141 | 1.026 | 0.950 | 0.892 | 0.847 |
| B | 0 | 0.004 | 0.037 | 0.157 | 0.338 | 0.473 | 0.582 | 0.675 |
| A | 0.809 | 0.776 | 0.749 | 0.725 | 0.703 | 0.683 | 0.664 | 0.648 |
| B | 0.755 | 0.826 | 0.893 | 0.955 | 1.014 | 1.064 | 1.115 | 1.165 |
| A | 0.633 | 0.619 | 0.605 | 0.515 | 0.464 | 0.378 | 0.279 | |
| B | 1.211 | 1.257 | 1.299 | 1.336 | 1.885 | 2.348 | 3.393 | |

In Figure 5 is shown curve $W_2$ and also those regions of the A, B plane where the segment of rest is stable for small displacements (region of conditional stability $V_1$) and where it is unstable for small displacements (region of absolute instability $V_2$).

The following paragraphs will reveal the meaning of these names (absolute and conditional) for the regions $V_1$ and $V_2$.



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## VI. FIELD OF ABSOLUTE INSTABILITY (REGION $V_2$)

In this case the qualitative picture of the transformation of the $\xi, \eta$ plane into itself is established very simply.

Let us study the auxiliary function

$$\varphi(t) = \frac{e^{-at} f(-t)}{f(t)}, \quad 0 < t \leq \theta, \tag{27}$$

which gives at least the sign of the ratio of the abscissae of the initial and the transformed points.

Function $\varphi(t)$ suffers an infinite discontinuity for that value of $t = \bar{t}$ for which $f(t) = 0$; whereupon the function changes its sign upon t's passing through the point of discontinuity. We have

$$\frac{d\varphi}{dt} = \frac{a}{1+b^2} \cdot \frac{e^{-at}}{[f(t)]^2} \{ e^{-(a+b)t} [\cos t + (a+b) \sin t] +$$

$$+ e^{(a+b)t} [\cos t - (a+b) \sin t] - 2 + [1 + (a+b)^2] \sin^2 t \} < 0.$$

Actually, denoting for sake of brevity the expression in irregular parenthesis by $\Phi(t)$, we get

$$\Phi'(t) = -[1 + (a+b)^2] 2 \sin t [\cos hyp (a+b)t - \cos t].$$

It means that $\Phi'(t) < 0$ when $0 < t < \pi$ and $\Phi'(t) > 0$ when $\pi < t < \theta \, (< 2\pi)$. Consequently, $\Phi(t)$ becomes a maximum at the ends of the interval. But, as is easily found: $\Phi(0) = \Phi(\theta) = 0$. Consequently, $\Phi(t) < 0$ and therefore $d/dt \, \varphi(t) < 0$ (Q. E. D.).

If we realize that $\lim_{t \to 0} \varphi(t) = -1$ and $\varphi(\theta) = e^{b\theta} \{ b \sin \theta - \cos \theta \} > 1$ (by virtue of the assumption of instability of the segment of rest), then it is clear that the graph of $\varphi(t)$ has the form schematically shown in Figure 6. From this it follows that in the case of instability of the segment of rest the following inequality holds $|\xi_1| > |\xi_0|$. However, this still does not solve the problem concerning the behavior of the points during repetition of transformation T, since for small $\xi$ the ration $\left|\frac{\xi_1}{\xi_0}\right|$ can approach as close to 1 as desirable. To do this let us examine the result of repetition of transformation: the point with abscissa $\xi_0$ is transformed by transformation $S^+$ to a point with abscissa $\xi_1$ and then by a transformation $S^-$ (and, perhaps, after intermediate transformation that does not change the abscissa), into a point with abscissa $\xi_2$.

We will show that $\left|\frac{\xi_2}{\xi_0}\right| > q > 1$, where q is a certain constant relative to $\xi_0$. To do this we notice that the points of ray $\lambda(\pi)$ transform into points of continuation of this ray; that is $k(\pi) = k(-\pi)$ (compare the finding of a fixed point of an infinitely removed straight line in Section IX, Part II). Consequently, if for a transformation $S^+$ the transition time is $t < \pi$, then for a transformed point the transition time by a transformation $S^-$ will be $t_1 > \pi$ (it can be shown that both transition times will $> \pi$). But by the nature of the function $\varphi(t)$ it is clear that when $t > \pi$ we have:

$$|\varphi(t)| > \min(|\varphi(\pi)|, |\varphi(\theta)|).$$

Denoting $\min(|\varphi(\pi)|, |\varphi(\theta)|) = q$, we obtain:

$$\left|\frac{\xi_2}{\xi_0}\right| = \left|\frac{\xi_2}{\xi_1}\right| \cdot \left|\frac{\xi_1}{\xi_0}\right| > q,$$

which was to be proven.

But from this it follows that after 2k applications of the transformations $S^+$ and $S^-$ (and, perhaps, a certain number of the transformation E), we will have for the abscissa of the transformed point $\xi_{2k}$:

$$|\xi_{2k}| > q^k |\xi_0|.$$

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From this it follows that no matter what the initial point (as long as $\xi_0 \neq 0$), the transformed point moves off toward infinity for repeated transformations.

In that case, when $\xi_0 = 0$, we will have $\xi_1 \neq 0$. Actually, as was already pointed out, only when $t = \pi$ the transformed point lies on the continuation of the ray upon which the initial point lies. The ray that corresponds to $t = \pi$ does not coincide with $\eta$-axis, because $f(\pi) = -(1+b^2)[1+e^{-(\alpha+b)\pi}] \neq 0$.

From this it follows that the points of the $\eta$-axis also fall away toward infinity.

Thus the problem of the qualitative picture of the transformation of the $\xi, \eta$ plant into itself is completely solved in the given case and it is established that for all values of A and B in the region $V_2$, for any initial conditions, the regulator produces ever-increasing amplitudes of oscillation. (A continuation of this work will be placed in one of the future issues of the journal.)

[Figures follow.]

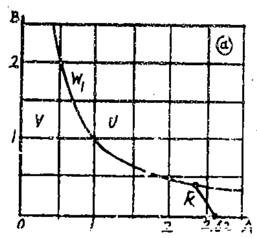


Figure 1

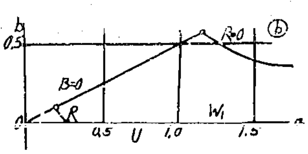


Figure 1a

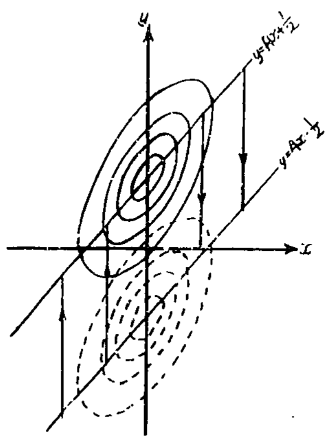


Figure 2



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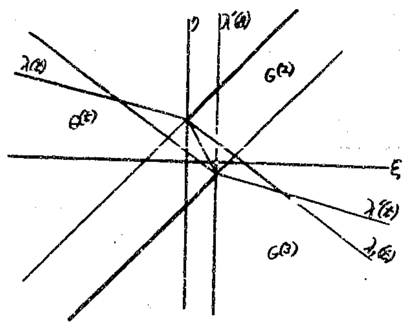


Figure 3

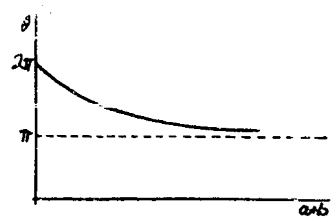


Figure 4

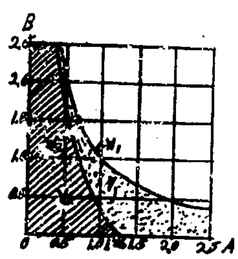


Figure 5

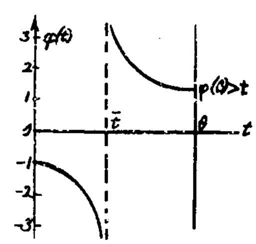


Figure 6





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